Optimal risk sharing in a collective defined
collection defined contribution pension system

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Abstract

This paper performs a simulation based study to define an optimal collective defined contribution pension system by considering an overlapping generations model of the pension system. The pensions rights are adjusted based on the returns earned by the fund portfolio by using a return smoothing mechanism to ensure the sustainability of the fund. The welfare provided by the pension payouts are compared in a utility based framework. We find that intergenerational risk sharing can smooth shocks over many generations even without creating a buffer. A desirable feature is the provision of smooth, highly auto-correlated indexation of pension rights. Depending on the discount factor considered, the optimal risk sharing implies that about one-fourth to one-third of underfunding should be passed on to younger cohorts in one year.

Keywords: Collective Defined Contribution, Funded pension system, Overlapping generations, Intergenerational risk sharing

JEL codes: G23

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1. Introduction

Current working generations and pensioners depend critically on the pension system for their retirement security. Moreover, pension funds have been in stress during the recent years due to several contributory factors including ageing population, longevity risk and more recently the financial crisis, the debt crisis and low long term interest rates. Keeping in mind these risks, pension funds already have or are increasingly moving from defined benefit (DB) to (collective) defined contribution (DC) types of pension systems (e.g. Ponds and Van Riel (2007) for Netherlands). This is also, in part, due to the maturity of pension plans since premium contributions can no longer be the tool to overcome underfunding. Instead, financial risk management and adjustment of pension rights based on financial market performance are becoming the main tools. Another reason for this switch is that the employers do not want spill over effects of underfunding on the company’s balance sheet and therefore they prefer the workers to be the risk bearers of their pensions. For participants this implies a shift in their risk exposures. If plans sponsors do not insure the funding of the system, participants can share the risk among themselves.

On one extreme of the pension systems we have individual pension accounts. However, it is known that individual households make poor choices in investing for their retirement (Choi et al. (2011)) and lack financial skills for retirement planning (Lusardi and Mitchelli (2007)). On the other extreme we have defined benefit systems. From theory, we know that the added value of DB is the intergenerational risk sharing where sponsors and new generations guarantee the liabilities to retirees. However, a DB system is expensive and increasingly unsustainable. A collective DC pension system with risk sharing mechanism can partially substitute for problems in both these extremes. This raises an important question of what might be an ideal collective DC system. In this paper we investigate a collective DC pension system in which we analyze to what extent risk sharing is desirable and what are the implications of a pension plan change to col-

\footnote{For example, Broeders and Rijksbergen (2010) point out that doubling the contributions would increase the funding ratio by less than 4% in the Dutch occupational pension system.}
lective DC on the fund’s assets, liabilities and also importantly future pension payments. We analyze the effects of various assumptions, in particular the discount rate for the future pension payments, on the introduction of such a system. The reason that discount rate has important implications is that the discounted sum of expected utility can have substantially different values when using different discount factors. This contributes to a major complication in the design of a new pension system.

To evaluate the benefits of such a funded collective DC schemes compared to individual defined contribution schemes a simulation based model is helpful. We numerically optimize over risk sharing parameter an overlapping generations model to understand to what extent risk sharing is desirable in this pension system. In our model, the assets and the liabilities of the fund evolve according to different factors. The assets are a function of market returns whereas the liabilities are dependent on the funding ratio of the fund. The way by which the pension rights (and hence the liabilities) depend on the funding ratio is called return adjustment mechanism (RAM). Since the liabilities would be adjusted based on the funding level (which in turn depends on investment risks), a smoothing parameter is used for risk sharing across cohorts. This parameter determines the extent of intergenerational risk sharing. One of the objective is to find the optimal risk sharing parameter where the optimality of the scheme is judged by the discounted sum of expected utility generated by pension payouts of all generations.

We find that intergenerational risk sharing can smooth shocks over many generations even without creating a buffer. A desirable feature is the provision of smooth, highly auto-correlated indexation of pension rights. It spreads pension reductions across several years. Depending on the discount factor considered the optimal risk sharing implies that only about one-fourth to one-third of funding mismatch should be passed on to the cohorts in one year. The choice of discount factor is an important consideration when redesigning a pension system. Changing discount factor might induce some redistribution.

There exists a large body of literature on funded pension systems. A major strand of literature concerns the benefits of intergenerational risk sharing in fully funded and collective pension systems. Beetsma and Bucciol (2011) investigate the introduction
of risk sharing in collective defined contribution plans and find it welfare increasing. Broeders et al. (2011) derive an analytic formula of liabilities in hybrid pension plan using market consistent valuation in a run-off scenario. However, we consider an ongoing system in which cohorts keep on entering and exiting the pension fund. This helps us to understand the behavior of the fund in the long-run and check the sustainability of the model. Cui et al. (2011) argue that collective pension schemes can be welfare enhancing compared to optimal individual investments because of efficient risk sharing. Gollier (2008) finds that collective pension schemes make it socially efficient to raise the collective risk exposure to take advantage of the equity premium with introduction of intergenerational risk sharing. Our work adds value in the following two ways. Firstly, most of the studies do not study time varying liabilities which depend on returns earned on the fund portfolio. Secondly we discuss how the optimal risk-sharing depends on preferences and parameters of the model. As expected, we find that discount factor is the most important parameter to consider when designing and evaluating pension schemes.

The analysis and results of this paper are relevant in practice. For example, the occupational pension funds in the Netherlands are considering switching from defined benefit average-wage schemes with solvency contingent indexation to collective DC with benefit adjustments. For a good overview of the Dutch pension system we refer to the relevant sections of Van Rooij et al. (2007) and Ponds and Riel (2009). The main reason for this switch is that to make the pension system sustainable in the long-run, pension funds need to be able to adjust the liabilities. Here, the return adjustment mechanism is very useful. Moreover, by using a return adjustment mechanism, the pension fund can automatically maintain a buffer and a significant funding shortfall is not passed on to future entrants. This improves the stability of the pension system. This makes the pension system sustainable in the long run. Another advantage is that the fund can keep the benefits of intergenerational risk sharing.
2. Mathematical Framework

In this section we describe our stylized model with return adjustment mechanism. We consider the model from the perspective of a pension fund with $N$ working and $K$ retired generations in an overlapping generations economy. The aim from perspective of the pension fund is to introduce a new collective DC pension system. Each year a generation exits and a young generation is added to the system. After $N$ years, when the worker retires, he is paid an variable annuity with payments that depend on indexation level. All the generations are assumed to have equal number of workers and the pension scheme has mandatory enrollment.

2.1. Assets of the fund

The asset side of the balance sheet comprises of the assets of the last year plus the returns earned on them and the contributions collected in the year. $C_t(\tau)$ denotes the yearly contribution at time $t$ paid by the working cohort that retires at time $\tau$. We assume that for $\tau = 1, \ldots, \infty$,

$$C_t(\tau) = \begin{cases} 1 & t = \tau - N, \ldots, \tau - 1, \\ 0 & \text{otherwise} \end{cases}$$

The pension fund has a choice to invest in a portfolio of risk free and risky assets. The return on the risky assets at time $t$ is denoted by $R_t^S = \exp(r_t^S)$; $r_t^S$ is assumed to be independent and identically normally distributed (N($\mu, \sigma^2$)). The risk-free interest rate is assumed constant and denoted by $R_f = \exp(r_f)$. The share in the risky-asset is denoted by $\omega$ which is assumed to be constant. We denote the financial assets of the fund at the end of the period $t$ by $A_t$. At the start of the year, the fund collects contributions $C_t = \sum_\tau C_t(\tau)$ and pays benefits of a total of $X_t$ to the the retired generations. After paying out the benefits and collecting the contributions from the working generations, assets are invested in the financial market. Therefore for $t = 1, 2, \ldots, \infty$,

$$A_{t+1} = R_t (A_t - X_t + C_t) \quad (1)$$

$$R_t = (1 - \omega)R_f + \omega R_t^S \quad (2)$$
2.2. Liabilities of the fund

The liability part consists of two entries. One is the pension rights of the working generations and the second is the remaining pension rights of the retired generations or annuitants. The pension rights of the individual at time $t$ retiring at time $\tau$ are denoted by $Z_t(\tau)$. The liabilities are the sum of pension rights of all individuals in the system. The pension rights of the retirees are decreased every year as they are paid annuity. The pension rights of the working and retired generations are indexed according to the indexation policy that is equal for all cohorts. Thus,

$$Z_{t+1}(\tau) = \begin{cases} 
(Z_t(\tau) + C_{t+1}(\tau))I_t & \text{if } \tau = t + 1, \ldots, t + N, \\
(Z_t(\tau) - X_t(\tau))I_t & \text{if } \tau = t, \ldots, t - K + 1,
\end{cases} \quad (3)$$

$$L_t = \sum_{\tau = t-N}^{t} Z_t(\tau) + \sum_{\tau = t+1}^{t+K-1} Z_t(\tau) = \sum_{\tau = t-N}^{t+K-1} Z_t(\tau)$$

here $X_t(\tau)$ is the payment at time $t$ to the generation retiring at $\tau$ and $L_t$ is the total liability of the pension fund at time $t$.

The liabilities include the rights earned so far by being in the pension system by the current working generations and the remaining rights of the retired generations. It does not include the present value of the future increase in the pension rights of both working and retired generations. Valuing liabilities in this way is similar to the actuarial concept of Accumulated Benefit Obligation (ABO) liabilities.

2.3. Indexation policy

The pensions rights are adjusted based on the returns earned by the fund portfolio by using the return adjustment mechanism to ensure the sustainability of the fund. This includes working as well and retired generations. For working generations this means that they receive positive and sometimes negative adjustments to their pension rights depending on the returns earned by the fund portfolio whereas the retired generations get annuity adjustments. The indexation rule generates mean reversion in the funding ratio of the fund.
Figure 1: Top Panel: Effect of the parameter $\theta$ on the indexation policy. The fund starts providing positive indexation when the funding ratio is above $\theta$. Here the $\alpha$ is fix at 0.25. Bottom Panel: Effect of the parameter $\alpha$ on the indexation policy. All the lines meet at $\theta$. At $\alpha = 1$, the slope is 45°. There is no floor hence the indexation policy is symmetric in funding ratio. Here the $\theta$ is fix at 1.
\[ i_t = \alpha \left( \ln \left( \frac{A_t}{L_t} \right) - \ln \theta \right) \] \tag{4}

The parameter \( \alpha \) is used for smoothing the shocks and \( \theta \) for specifying the funding ratio level above which positive indexation starts. Thus, the parameter \( \alpha \) determines the extent to which the intergenerational risk sharing is allowed and the parameter \( \theta \) specifies the level of funding ratio in the long run. The indexation on the pension rights is denoted by \( I_t = \exp(i_t) \).

Payouts are paid as a level annuity for \( K \) years. It is easy to see that if annuity rate = \( I_t^{-1} \) we get a level annuity. \( X_t \) denotes the sum of all payments by the fund in year \( t \), \( n \) is the number of retired generations then, it is easy to see that

\[
\frac{X_{t+1}}{X_t} = \left( \frac{1 - I_{t+1}^{-1}}{1 - I_t^{-1}} \right) \left( \frac{1 - I_t^{-(n-1)}}{1 - I_{t+1}^{-(n-1)}} \right)
\]

In the above expression, if \( I_{t+1} \) is equal to \( I_t \) then then the ratio of \( X_{t+1} \) to \( X_t \) simplifies to one. If \( I_{t+1} \) is greater than \( I_t \), then the ratio is also greater than one and if \( I_{t+1} \) is less than \( I_t \), then the ratio is less than one. This makes intuitive sense, as payments follow the pattern of indexation.

It is clear that the return on assets on a given year impacts the indexation rule for following years by way of smoothing the pension returns. The indexation rule provides a stable indexation compared to volatile asset return. This is a highly desirable feature in a pension fund as retirees do not have to adjust their consumption levels because of the volatility of the stock market. Solvency of the pension fund is not an issue here since the pre-determined indexation rule can provide negative indexation in ‘bad times’ to return the funding ratio to a healthy level as determined by the parameter \( \theta \).

3. Steady state

In this section, we will consider the steady state of the pension fund, that is when there is no specific drift over time in the variables of the model e.g. payouts or funding ratio. When the pension plan is redesigned, choosing arbitrary model parameters would imply that the some generations are better off than others. We ignore the time period it
takes to reach the steady state, i.e. the transition period, and only consider the steady state. Thus, we ignore the welfare costs associated with the redesign, however, steady states can still help us intuitively understand the optimal design parameters. Thus we aim to provide some intuitive explanations of how the model parameters effect e.g. the level of indexation and payouts in the long run.

3.1. Deterministic steady state

In the non-stochastic stationary state all contributions and payouts are constant over time. In addition we require assets and liabilities to remain constant. The first steady state that we consider is a pension plan where each cohort has its individual account on which it accumulates wealth. The pension system provides the same return $R$ in every period to all cohorts. The pension fund in this case is nothing but the aggregate of all individual accounts.

In this fund the assets are equal to the liabilities. All the fund does is collect contributions, invest them at a risk free rate and distribute the returns as an annuity. There is no redistribution among cohorts.

To define the stationary quantities, let $Z_i$ be the pension wealth of a cohort that has been participating in the fund for $i$ years. Wealth is accumulated during the first $N$ years and decumulated in the final $K$ years. During the accumulation stage wealth evolves as

$$Z_{i+1} = R(Z_i + C)$$

where $C$ is the (fixed) contribution. Taking into account the initial condition $Z_0 = 0$ wealth for each cohort is

$$Z_i = CR \frac{R^k - 1}{R - 1} \quad i = 1, 2, \ldots, N \quad (5)$$

After retirement the pension wealth is distributed as an annuity payout

$$X = Z_N \frac{1 - 1/R}{1 - 1/R^K} = CR^N - 1 \quad \frac{1}{1 - R^{-K}}$$
Remaining wealth at each period after retirement is

$$Z_{N+i} = Z_N \frac{1 - R^{i-K}}{1 - R^{-K}} = C R^N \frac{1 - 1 - R^{i-K}}{R - 1}$$

(6)

using (5) and (6), Total assets, which are equal to liabilities, for the fund are

$$A = L = \sum_{i=1}^{N+K} Z_i = C \frac{R}{R - 1} \left( K \frac{R^N - 1}{1 - R^{-K}} - N \right)$$

(7)

The assets satisfy the fixed point property

$$A = R(A - KX + NC)$$

(8)

One can interpret the fund either as an individual defined contribution scheme or as a defined benefit scheme. Without risk there is no difference. When returns are risky, however, the risk sharing matters. To remain a defined benefit scheme the sponsor has to compensate the return differential

$$D_{t+1} = (R - R_{t+1})(A + NC - KX)$$

(9)

When it does, assets, contributions, and benefits remain at their constant levels. Many more stationary states exist at which the coverage ratio is not equal to one. If the fund has a buffer, it earns a return on this buffer that it can use to finance pension payments above what is earned on the contributions. The only condition for stationarity is (8). For a general payout level X this can be written as

$$A = \frac{R}{R - 1} (KX - NC)$$

(10)

which defines the level of assets that can support the payments X. The fixed point condition does not indicate where the additional assets come from. They must either be paid in a transition period by cohorts that will receive less benefits or pay more contributions, or they have been a birth gift at the start of the fund. We consider an
adjustment mechanism to converge to the steady state from any given initial condition. It defines an indexation rate $I$, possibly different from $R$, to define the pension accumulation and annuity levels,

$$X = C \frac{I^N - 1}{1 - I^{-K}}$$ (11)

$$L = C \frac{I}{I - 1} \left( \frac{I^N - 1}{1 - I^{-K}} - N \right)$$ (12)

$$I = \left( \frac{A}{\theta L} \right)^\alpha$$ (13)

The system (8) to (13) has 4 equations in the four unknowns $A$, $X$, $L$ and $I$. To characterize the solution rewrite (13) as

$$A = \theta L I^{1/\alpha}$$

and divide by (10) to obtain

$$1 = \theta I^{1/\alpha} \frac{I}{R} \frac{R - 1}{I - 1}$$ (14)

as a nonlinear equation in $I$ given the return $R$ and the adjustment parameters $\alpha$ and $\theta$.

Figure 1 shows the stationary solutions for different values of $R$ and a few selected parameter combinations ($\theta, \alpha$) The combination of a high value for $\theta$ like $\theta = 1$, and slow adjustments leads to a steady state in which indexation is much larger than the financial return. This solution requires a correspondingly high coverage ratio. With either a lower value of $\theta$ or a faster adjustment speed $\alpha$ the steady state solution varies a little bit with $R$. The parameters have much more impact on the stationary values for the coverage ratio. With $\theta = 1$ the steady state for $A/L$ will always be above one. For $\theta < 1$ it depends on the financial return $R$ whether the pension system will exhibit over- or underfunding. Since these are steady states, the underfunding is sustainable, but each cohort would be better off with an individual account.

A particularly interesting solution obtains if $\theta = R^{-1/\alpha}$. In this case $I = R$ and the coverage ratio will always be equal to one for any value of $\alpha$. This suggests that we should decrease the value of $\theta$ when we increase the degree of return smoothing.
The optimal parameters for the steady state is the combination \((\theta, \alpha)\) that maximizes the payout \(X\). Since payout is increasing in \(I\), the largest possible value for \(I\) is optimal. We can find a large \(I\) by setting \(\alpha\) as small as possible and \(\theta\) as large as possible. Without a restriction there is no interior optimum for \((\theta, \alpha)\). We need a restriction, since the different solution are not comparable. Large value of \(\theta\) imply that much of the payouts come from returns on the buffer. Of course, by increasing the pension assets \(A\), we get higher payouts \(X\). For a fair comparison we should therefore only compare solutions with the same level of assets. That means we need the budget constraint

\[ A = \bar{A} \]

Admissible values for \(\alpha\) and \(\theta\) must satisfy the two constraints

\[ I = \left( \frac{\bar{A}}{\theta L} \right)^\alpha \]

\[ \bar{A} = \frac{R}{R - 1} (KX - NC) \] (15)

with \(X\) and \(L\) still given by (11) and (12) as a function of \(I\). Since the fixed point constraint (15) does not depend on \(\alpha\) and \(\theta\), this fully determines \(I\). The maximum \(I\) is therefore independent of the pension design parameters \(\alpha\) and \(\theta\) in a steady state.

3.2. Stochastic steady state

In this section we consider the stochastic steady state of the model. Therefore, we no longer assume that the portfolio returns are fixed but that the risky returns are log-normally distributed. We assume that the parameters of the log-normal distribution are \(\mu = 6\%\) and \(\sigma = 15\%\), the share in the risky asset is fix at \(\omega = 0.6\) and risk-free rate is \(R^f = 2\%\). We take \(N = 40\) and \(K = 15\) and thus we have 55 overlapping generations. For tables 1 and 2 the number of simulations = 100,000. In the tables, we show the value of various pension fund indicators for different values of choice parameters \(\alpha\) and \(\theta\).

From table 1, we can see that the confidence interval of indexation is much smaller than that of the returns. This shows the smoothing of risk across generations implies
Figure 2: The stationary solution for the indexation level I (top panel) and the funded ratio $A/L$ (bottom panel) as a function of the financial return $R$. Parameters are ($\alpha = 0.2, \theta = 1$) for the solid line, ($\alpha = 0.2, \theta = 0.9$) for the dotted line, and ($\alpha = 0.4, \theta = 1$) for the dashed line.
less uncertainty of indexation in pensions. For $\alpha = 0.2$, the interval is smaller than that of $\alpha = 0.4$ both for a given $\theta = 1$. Thus more risk sharing implies less uncertainty of indexation. However, also from the table 1 we see that the funding ratio in steady state for $\alpha = 0.2$ is higher than that of $\alpha = 0.4$. This shows that the fund requires more surplus to smooth the risks over a long horizon. The payouts are also higher for lower alpha because the return earned on the extra surplus maintained to smooth shocks is also distributed to the working and retired generations. However, there are costs associated with saving for this buffer as the generations retiring just after pension plan redesign are indexed less to save for buffer. The parameter $\theta$ then is helpful as it determines the level of funding ratio in long run. We can choose the parameter $\theta$ to arrive at a steady funding level that is suitable. For a given value of $\alpha$, lower $\theta$ implies lower funding ratio in the long run. We also see that the indexation is highly autocorrelated and the autocorrelation increases as we decrease $\alpha$ compared to the returns which are not at all autocorrelated. Thus we find that the main benefit of having intergenerational risk-sharing is that the indexation on pension is much smoother than the market returns and also highly correlated. This is a desirable feature in a pension system as it will provide time for individuals to smooth their consumption levels.

In table 2, we make the parameter $\theta$ a function of $\alpha$ so that the funding ratio on an average remains equal to one and there is no surplus creation. Thus table 2 shows a funding ratio of approximately 1. We still observe that the confidence interval of indexation is lower than that of returns thus illustrating smoothing. We also observe in the table 2 that for a given $\theta$, lower $\alpha$ actually implies a more volatile funding ratio. This is because the fund does not quickly adjust to the mismatch in steady state funding ratio with a lower value of $\alpha$ but delays the shock over a long horizon. This has implications for generations entering the fund which shows the trade-off between too much and too little risk sharing. Thus the main result of this subsection is that there is a trade-off between uncertainty in indexation and uncertainty in funding ratio.

Figure 3 shows the kernel estimate of the distribution of funding ratio when the fund has reached its steady state. The value of the function at funded ratio 1 shows the probability of underfunding of the pension fund. We see that lower $\alpha$ implies less chances
of underfunding. The chances increase as we increase the risk sharing parameter $\alpha$. The distribution of funding ratio with $\alpha = 0.25$ stochastically dominates the distribution of funding ratios with higher $\alpha$'s for given fixed $\theta$. Thus $F_{\alpha=0.25}(x) \leq F_{\alpha=0.5}(x)$. Therefore, when comparing only the steady state's funding ratios, we would always choose the lowest $\alpha$ and hence most risk sharing. As a proxy of stability and sustainability of the pension fund, this figure shows the lowest $\alpha$ provides the most stability. This is due to the fact that the lower $\alpha$ implies a high buffer. However, as was mentioned before, there are costs associated by way of creation of buffer of reaching this state.

4. Results

Here we present the results of simulation where the fund does not start in a steady state. Instead the fund is built up from individual account and then then intergenerational risk sharing is introduced. This helps us to understand what happens when the
Table 1: Pension fund indicators in steady state

<table>
<thead>
<tr>
<th>Panel</th>
<th>Mean</th>
<th>95 percentile</th>
<th>5 percentile</th>
<th>Std. error</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. $\alpha = 0.2$ and $\theta = 0.9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indexation</td>
<td>1.048</td>
<td>1.109</td>
<td>0.991</td>
<td>0.0001</td>
<td>0.7506</td>
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<tr>
<td>Funding ratio</td>
<td>1.511</td>
<td>1.512</td>
<td>0.861</td>
<td>0.0006</td>
<td>0.7467</td>
</tr>
<tr>
<td>Payouts</td>
<td>196.0</td>
<td>441.0</td>
<td>69.0</td>
<td>0.4338</td>
<td>0.9008</td>
</tr>
<tr>
<td>Assets</td>
<td>3610.0</td>
<td>7660.0</td>
<td>1420.0</td>
<td>7.1126</td>
<td>0.8788</td>
</tr>
<tr>
<td>Liabilities</td>
<td>3040.0</td>
<td>5830.0</td>
<td>1460.0</td>
<td>4.7940</td>
<td>0.9461</td>
</tr>
<tr>
<td>B. $\alpha = 0.2$ and $\theta = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indexation</td>
<td>1.056</td>
<td>1.119</td>
<td>0.997</td>
<td>0.0001</td>
<td>0.7522</td>
</tr>
<tr>
<td>Funding ratio</td>
<td>1.328</td>
<td>1.758</td>
<td>0.986</td>
<td>0.0008</td>
<td>0.7484</td>
</tr>
<tr>
<td>Payouts</td>
<td>259.0</td>
<td>622.0</td>
<td>78.0</td>
<td>0.6628</td>
<td>0.9014</td>
</tr>
<tr>
<td>Assets</td>
<td>5080.0</td>
<td>11460.0</td>
<td>1790.0</td>
<td>11.5110</td>
<td>0.8785</td>
</tr>
<tr>
<td>Liabilities</td>
<td>3680.0</td>
<td>7490.0</td>
<td>1610.0</td>
<td>6.6745</td>
<td>0.9457</td>
</tr>
<tr>
<td>C. $\alpha = 0.4$ and $\theta = 0.9$</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Indexation</td>
<td>1.043</td>
<td>1.128</td>
<td>0.965</td>
<td>0.0002</td>
<td>0.5718</td>
</tr>
<tr>
<td>Funding ratio</td>
<td>1.004</td>
<td>1.217</td>
<td>0.8240</td>
<td>0.0004</td>
<td>0.5686</td>
</tr>
<tr>
<td>Payouts</td>
<td>154.0</td>
<td>312.0</td>
<td>63.0</td>
<td>0.2660</td>
<td>0.8040</td>
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<tr>
<td>Assets</td>
<td>2640.0</td>
<td>4840.0</td>
<td>1280.0</td>
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<td>0.8667</td>
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<tr>
<td>Liabilities</td>
<td>2610.0</td>
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<td>1380.0</td>
<td>3.3170</td>
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<td></td>
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</tr>
<tr>
<td>Indexation</td>
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<td>1.135</td>
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<td>Funding ratio</td>
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<td>1.373</td>
<td>0.925</td>
<td>0.0004</td>
<td>0.5740</td>
</tr>
<tr>
<td>Payouts</td>
<td>186.0</td>
<td>392.0</td>
<td>71.0</td>
<td>0.3498</td>
<td>0.8083</td>
</tr>
<tr>
<td>Assets</td>
<td>3370.0</td>
<td>6430.0</td>
<td>1550.0</td>
<td>5.1819</td>
<td>0.8676</td>
</tr>
<tr>
<td>Liabilities</td>
<td>2960.0</td>
<td>5370.0</td>
<td>1480.0</td>
<td>4.0889</td>
<td>0.9389</td>
</tr>
</tbody>
</table>

Note: This table shows the mean, 95 and 5 percentiles, standard error = $\sigma/\sqrt{\text{no. of simulations}}$, and autocorrelation with lag 1 (autocorrelation for lag zero =1) for various variables of pension fund in steady state (here at time point 200 after start) for different parameter values of $\alpha$ and $\theta$. The values have been rounded off according to the standard errors. Number of simulations are 100,000. The mean return for each of the simulations is 1.046, with 95th percentile = 1.215 and 5th percentile = 0.901, standard error of 0.0003 and autocorrelation of -0.0192. The parameters of the log-normal distribution are $\mu = 6\%$ and $\sigma = 15\%$; the share in the risky asset is fix at $\omega = 0.6$ and risk-free rate is $R_f = 2\%$.
Table 2: Pension fund indicators in steady state with endogenously determined $\theta$

<table>
<thead>
<tr>
<th>Panel A. $\alpha = 0.2$ and $\theta = 0.8018$</th>
<th>Mean</th>
<th>95 percentile</th>
<th>5 percentile</th>
<th>Std. error</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexation</td>
<td>1.041</td>
<td>1.101</td>
<td>0.986</td>
<td>0.0001</td>
<td>0.7498</td>
</tr>
<tr>
<td>Funding ratio</td>
<td>0.992</td>
<td>1.296</td>
<td>0.746</td>
<td>0.0005</td>
<td>0.7458</td>
</tr>
<tr>
<td>Payouts</td>
<td>153.0</td>
<td>326.0</td>
<td>61.0</td>
<td>0.2970</td>
<td>0.9002</td>
</tr>
<tr>
<td>Assets</td>
<td>2620.0</td>
<td>5270.0</td>
<td>1130.0</td>
<td>4.5712</td>
<td>0.8791</td>
</tr>
<tr>
<td>Liabilities</td>
<td>2580.0</td>
<td>4690.0</td>
<td>1340.0</td>
<td>3.5900</td>
<td>0.9466</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. $\alpha = 0.4$ and $\theta = 0.8954$</th>
<th>Mean</th>
<th>95 percentile</th>
<th>5 percentile</th>
<th>Std. error</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indexation</td>
<td>1.042</td>
<td>1.128</td>
<td>0.965</td>
<td>0.0002</td>
<td>0.5715</td>
</tr>
<tr>
<td>Funding ratio</td>
<td>0.998</td>
<td>1.210</td>
<td>0.819</td>
<td>0.0004</td>
<td>0.5683</td>
</tr>
<tr>
<td>Payouts</td>
<td>153.0</td>
<td>308.0</td>
<td>63.0</td>
<td>0.2627</td>
<td>0.8038</td>
</tr>
<tr>
<td>Assets</td>
<td>2610.0</td>
<td>4780.0</td>
<td>1270.0</td>
<td>3.6789</td>
<td>0.8666</td>
</tr>
<tr>
<td>Liabilities</td>
<td>2590.0</td>
<td>4530.0</td>
<td>1380.0</td>
<td>3.2857</td>
<td>0.9389</td>
</tr>
</tbody>
</table>

Note: This table shows the mean, 95 and 5 percentiles, standard error = $\sigma/\sqrt{\text{no. of simulations}}$ of simulations and autocorrelation with lag 1 (autocorrelation for lag zero = 1) for various variables of pension fund in steady state (here at time point 200 after start) for different parameter values of $\alpha$ and $\theta = \exp(-\mu_p/\alpha)$. The values have been rounded off according to the standard errors. Number of simulations are 100,000. The parameters of the log-normal distribution are $\mu = 6\%$ and $\sigma = 15\%$; the share in the risky asset is fix at $\omega = 0.6$ and risk-free rate is $R_f = 2\%$

Accrued liabilities of the fund are not changed on pension fund redesign and the fund does not start in the steady state state. We can thus also check if the pension fund should have no buffer and maintain a funding ratio of 1 or is it optimal to have a smoothing buffer to ensure better intergenerational risk sharing.

4.1. Initial allocation

One important consideration is the initial funding ratio, since it determines the starting indexation of the pension system. We consider the situation when it is one. The collective defined contribution pension fund is made by merging individual accounts of $N$ working and $K$ retired generations. The $N$ working generations (one generation for every birth cohort) have been saving separately for their retirement by investing a fixed amount and the share in the risky asset is fixed. The market return that the assets receive is equal to the expected return that the assets will receive after the new pension fund has been created. The annuitants are paid annuity with annuity rate equal to the expected returns. The assets in the first time period are therefore the sum of all the individual accounts. The liabilities of the pension fund to the individual cohort is also
equal to the assets in the their pension account before merging. In this way we start with a funding ratio of 1. Later on, we experiment with various other starting initial funding ratios where the excess buffer is considered as a birth gift to the fund.

Thus, our model starts at time \( t = -(N + K) \) when the first generation sets up their individual account for retirement. The retirement date is denoted by \( \tau \). At time \( t = -(N + K - 1) \), we add another generation which sets up its own individual account for retirement. Thus, we then have two accounts for two generations. Proceeding in a similar way, we have \( N \) generations with \( N \) individual accounts at time \( t = -(K + 1) \). At this time \( t = -K \) we have a retired generation. Proceeding in the similar way at time \( t = -1 \), we have \( N \) working and \( K \) retired generations. At that time, the social planner sets up a collective defined-contribution pension fund by merging individual pension accounts. The value of assets \( A_0 \) at time \( t = 0 \) of the new pension fund is thus the sum of value of all individual accounts. The initial liabilities of the pension fund to each cohort are exactly equal to the value of their assets before merging. The total liabilities are thus the sum of the individual liabilities of each cohort. In this way, \( L_0 = A_0 \) and the starting funding ratio of the new pension fund is 100%. In this way the starting values are independent of the model parameters \( \theta \) and \( \alpha \). We experiment with different initial funding ratio by taking \( A_0 = \text{Funding ratio} \times L_0 \). This means that we assume that the fund receives a birth gift.

4.2. Simulations

We run 10,000 simulations for 250 years starting with the initial conditions mentioned in the last section. We experiment with different values of risk sharing parameter \( \alpha \) (0.25, 0.50, 0.75, 1) and initial funded ratios (100%, 110%, 120%, 130%). As is usually the case in practice, we take the share in the risky asset \( \omega \) constant at 60%.

Top panel figure 4 shows the expected benefit payouts given by the pension fund over 250 years. We see that for smallest \( \alpha \) the expected benefits in the few initial years in lowest, however, they become highest in the later years. The reason is that when the \( \alpha \) is low the pension fund is building up buffer for the future generations which is used for risk sharing. The effect is most pronounced for a low \( \alpha = 0.25 \) which corresponds to most risk sharing.
Figure 4: Top Panel: Expected pension benefits with different risk sharing parameter $\alpha$. $\mu = 6\%$ and $\sigma = 15\%$, risk-free rate is $R_f = 2\%$, Number of simulation=10000, $\theta = 1$, $\omega = 0.6$ Bottom Panel: Expected benefit payouts by the pension fund with different initial funding ratios. $\mu = 6\%$ and $\sigma = 15\%$, risk-free rate is $R_f = 2\%$, Number of simulation=10000, $\theta = 1$, $\alpha = 0.5$ $\omega = 0.6$
The graph is smoother for \( \alpha = 0.25 \) than for \( \alpha = 1 \) illustrating smoothing by the risk-sharing parameter. Next, we fix \( \alpha = 0.25 \) and compare the benefits for different initial funded ratios. Bottom panel of figure 4 shows the mean benefit payouts for 250 years starting with different initial funding ratios. The benefits are higher to generations retiring earlier when the pension fund had more money at the start (at \( t = 0 \)) compared to the case when the funded ratio is 1. We also observe that the fund does not start in the steady state but eventually reaches it after some years. Implying that the the steady position does not depend on the initial funded ratio but on model parameters \( \alpha \) and \( \theta \). We can also see that the disadvantage of building are absorbed by generations retiring just after the pension plan change. The main result of this subsection is that it is easier to start risk sharing when there is a buffer available. Since creation of buffer takes time and some generations receive indexation cuts to save for buffer creation. This might also have implications for choosing one set of parameters for funds starting at different funding levels.

4.3. Fund policy

Let \( X_t(\tau) \) be the payments in period \( t \) to a surviving cohort that has retired in period \( \tau \). Living generations are indexed by their retirement date and receive benefits until they die at time \( \tau + K \). In any period \( t \) the payments of the fund are represented by the \( K \)-vector

\[
X_t = (X_t(t - K + 1), \ldots, X_t(t))
\]

Retirees evaluate the benefits of these payments using the utility function \( U_t = U(X_t) \). The fund’s objective is the maximize the long-run utility of all current and future cohorts, the optimal risk-sharing parameter \( \alpha \) and parameter \( \theta \) is chosen such that they maximize the discounted sum of expected utility generated by pension wealth.

\[
V = \sum_{t=0}^{\infty} \delta^t \mathbb{E}_0[U_t]
\]

where \( \delta \) denotes the discount factor. The fund faces a budget constraint in its optimization, since contributions are fixed. The fund’s policies are its investment decisions and its distribution of benefits. The utility function is not necessarily separable in the
benefits of the individual cohorts. Generations may view their benefits from the system relative to what other cohorts receive at the same time. Such preferences induce a degree of fairness on the distribution of payments over different cohorts. Several utility functions express both the intratemporal and intertemporal effects of the benefits policies. One simple parametric formulation is

$$U(X_t) = \frac{X_t^{1-\gamma}}{1-\gamma}$$

where $X_t = \sum_{\tau=t-K+1}^{t} X_t(\tau)$ is the total payout in period $t$. To emphasize the preference for equality the average could be replaced by the CES specification

$$\bar{X}_t = \left( \frac{1}{\sum_{\tau} X_t(\tau)^\rho} \right)^{1/\rho}$$

where $\rho$ is the substitution elasticity. Like the total payout $X_t$ the CES function is first degree homogeneous, meaning the an overall proportional increase in payments to all cohorts raises the average by the same proportion. The utility function is still time separable, that is to say that the utility in one time period does not depend on the utility in another time period. When $\rho = 1$, it simplifies to constant relative risk aversion (CRRA).

4.4. Optimal risk sharing

The shape of the top panel of figure 4 makes it difficult to judge which of the $\alpha$ is better than the others if we compare the value generated by pension benefits according to equation (16) considering only the cases without the budget constraint of $\theta = \exp(-\mu/\alpha)$. The budget constraint makes sure that the funding ratio on average remains approximately 1 and thus there is no buffer creation. The best $\alpha$ is dependent on the discount factor ($\delta$) we choose. This is because lower $\delta$ will weigh generations retiring earlier more heavily than generations retiring later. As for example, we can always find a $\delta$ such that $\alpha = 0.25$ is best and also for $\alpha = 1$ we can find a $\delta$ such that it is the best. Smaller alpha translates into higher buffer and hence higher expected benefits in
Table 3: Optimal risk sharing parameter $\alpha$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Discount factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.96</td>
<td>0.97</td>
</tr>
<tr>
<td>0.97</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Panel A. Baseline results

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Discount factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.2243</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2312</td>
</tr>
<tr>
<td>1</td>
<td>0.2381</td>
</tr>
</tbody>
</table>

Panel B. $\sigma = 5\%$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Discount factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.2462</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2544</td>
</tr>
<tr>
<td>1</td>
<td>0.2628</td>
</tr>
</tbody>
</table>

Panel C. $\gamma = 3$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Discount factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.2335</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2397</td>
</tr>
<tr>
<td>1</td>
<td>0.2461</td>
</tr>
</tbody>
</table>

Note: This table shows optimal value of $\alpha$ when we numerically optimize over this parameter considering CES utility function with parameter (substitution elasticity) $\rho$; Other parameters: No. of simulations = 1000, $\theta = \exp(-\mu/\alpha)$, Risk free rate=2%, Mean= 6%, var=15%, T=250 years, $\omega = 0.60$. $\delta = \frac{1}{1+r}$ so when $\delta = 0.96$, $r = 0.0416$

later generations due to returns earned on the buffer$^3$. Therefore in table 3 we consider optimization of (16) over $\alpha$ with budget constraint.

From the panel A of table 3, we see that optimal value of risk sharing parameter generally falls between $\frac{1}{4}$ and $\frac{1}{3}$. Thus it is optimal to have the risk sharing such that in underfunding about one third or one fourth of funding mismatch is passed on to the individuals in the year. One trend we can observe from the table 3 is that the parameter $\alpha$ decreases as the discount factor decreases. Having a lower discount factor means individuals in the pension system care less about future generations. Thus they prefer lower $\alpha$ which means more risk sharing, less uncertainty in indexation and high auto-correlation but high uncertainty in funding ratio (see table 2). High uncertainty in funding ratio is not preferable for cohorts just entering the fund as they may enter in

$^3$In a set of unreported optimization of (16) jointly with respect to $\alpha$ and $\theta$, we find that the optimal parameter values create a large buffer which provides extra return and hence higher indexation in later years at the cost of lower indexation to generations retiring just after pension plan redesign. When optimizing jointly over $\alpha$, $\theta$ and $\omega$ we find that it is always optimal to have 100% in equity because of equity premium.
an underfunded pension system. The parameter $\rho$, which is the substitution elasticity, is used to emphasize equality across generations at a point in time. Less substitution elasticity implies it is optimal to have more risk sharing. There it is optimal to have more risk sharing if we care for for intra-generational equality.

We also check if the patterns we observed still hold under other parameter values. We also look closely how these different parameter values affect the optimal model parameter $\alpha$. We check for volatility in stock returns and risk aversion parameter which are shown in the panel B and C of the table 3. We find that the patterns are robust under different parameter settings. We lower the volatility from 15% to 5%. We see that risk sharing parameter $\alpha$ has increased. Thus lower volatility generally implies less risk sharing. This is because if there is less risk then higher risk sharing is not required. Similar pattern also holds for different relative risk aversion parameter.

To check how different are two choices of $\alpha$ from each other in optimization we calculate the certainty equivalents. Lets denote $V_{\text{max}}$ and $V$ as the utility generated
by optimal $\alpha$ and the given $\alpha$ respectively. Then we first solve for $K$ in the following equation and then for $c$.

$$V = \sum_{t=1}^{\infty} \delta^t (K)^{(1-\gamma)}/(1 - \gamma)$$

$$V_{\text{max}} = \sum_{t=1}^{\infty} \delta^t (cK)^{(1-\gamma)}/(1 - \gamma)$$

The value $c$ is plotted in the figure 5. The parameter values are $\gamma = 2$, discount factor $\rho = 0.97$, $\rho = 1$ and we calculate the sum till $t_{\text{max}} = 250$. We find that these values are economically significant with choosing a non-optimal $\alpha$ can cost more than 4% per annum.

5. Conclusion

This paper considers a realistic overlapping generations pension fund with 40 working and 15 retired generations in which the indexation of pension rights is dependent on the funding level of the pension fund. We analyze the introduction of a collective defined contribution pension system which allows intergenerational risk sharing. The welfare provided by the pension payouts are compared in a utility based framework. The utility functions take into account intra-generational fairness.

We find that intergenerational risk sharing can smooth shocks over many generations even without creating a buffer. A desirable feature is the provision of smooth, highly auto-correlated indexation of pension rights. It spreads pension reductions across several years. Depending on the discount factor considered the optimal risk sharing implies that only about one-fourth to one-third of underfunding should be passed on to the cohorts in one year. The choice of discount factor is an important consideration when redesigning a pension system. Changing discount factor might induce some redistribution. For example, risk sharing parameter might not be optimal for a different discount factor. This was the case, for example, in the Netherlands when the discount rate was changed in 2004 and more recently again in 2012.

In this article we have made some obvious simplifications. We model constant contributions. Instead, time varying contributions provide an extra degree of freedom which
can we used to stabilize underfunding. However, because of maturity of pension plans this might not be a very effective tool, therefore we do not model it. It is also straightforward to model a floor on the indexation policy, however we refrain from these added complexities to focus on the symmetric indexation rule.

References

Beetsma, R., Bucciol, A., 2011. Risk sharing in defined-contribution funded pension system.


